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A General Learning Theory and Its Application to the Acquisition of Proof Skills in Geometry

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Mathematics education	Proofs	Instantiation
Skill acquisition	Problem-solving	Analogy
Learning	Search	Generalization
Production system	Representation	Discrimination

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Generating a proof in geometry is analyzed into the stages of proof planning and proof generation. This report focuses on the processes of proof planning. In this stage the student searches for a proof tree which relates the givens to what is to be proven. This search is performed by operators that reason forward from the givens and that reason backwards from the to-be-proven statement. We discuss three types of learning which underlie this search. There is acquiring the operators in procedural form, learning to represent the problem in a way the operators can apply, and tuning the operators so that they will apply more appropriately.

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20. Abstract (Continued)

The paper focuses discussion on the first and third. The mechanisms underlying the first process are composition and instantiation. They transform information processing from interpretative application of procedural knowledge. The mechanisms underlying knowledge tuning are analogy, generalization, discrimination, and composition. They serve to add further constraints to the productions that embody these rules.

19. Key Words (Continued)

Proceduralization
Tuning
Automatization

Accession For
NTIS GRA&I
DDC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or special
A

Introduction

It would be a significant contribution if psychology could provide an explanation of how cognitive skills are acquired. We have been working on a general theory of learning called ACT. We have developed some learning mechanisms out of general considerations about human learning. Particularly important to us in this initial formulation phase were considerations of language acquisition. We have since attempted to validate this theory by applying it to various domains. Anderson, Kline, and Beasley (1979) describes a successful application of this theory to the literature on schema formation or prototype formation. The current report is concerned with an attempt to apply the learning theory to the acquisition of proof skills in geometry.

American psychology, in the middle third of this century, was dominated by a concern with learning theory. At least within the circles of cognitive psychology that effort to develop a learning theory has come into disrepute and the negative evaluation has generalized to some degree to cover any concern with learning theory. The reasons for the difficulty with traditional learning theory has been discussed at length (e.g., Anderson & Bower, 1973; Anderson, 1976; Chomsky, 1959; Bever, Fodor, & Garrett, 1968). The basic criticism was that the computational mechanisms (for example, S-R bonds) proposed in these theories were not of sufficient computational power to explain the sophistication of human behavior. It is interesting that these criticisms were really directed at the performance assumptions of the theories and not the learning assumptions.

The performance aspect of our theory, ACT, involves a production system (Newell, 1973; Newell & Simon, 1972) operating on a semantic network. The system is quite powerful computationally and also plausible psychologically. Therefore, it will not be subject to the kinds of criticisms that were directed at standard learning theory. ACT is a computer simulation program and virtually everything described has been implemented. While this does serve to establish the sufficiency of the theory, the emphasis in this discussion will be on the basic ideas and not on their implementations.

In this presentation we will be assuming some basic familiarity with properties of production systems although much should be self-evident even to the uninitiated. We will first discuss how the ACT model applies to planning, an important aspect of geometry proof generation. Then we will

discuss how the skills required for successful planning are acquired.

Planning

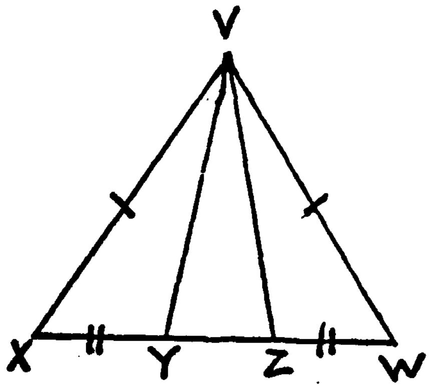
Most successful attempts at proof generation can be divided into two major episodes -- a period in which a student attempts to find a plan for the proof and a period in which the student translates that plan into an actual proof. The first stage we call *planning* and the second *execution*. It is true that actual proof generation behavior often involves alternation back and forth between the two modes -- with the student doing a little planning, writing some proof, running into trouble, planning some more, writing some more proof, and so on. Still we believe that planning exists as a logically and empirically separable stage of proof generation. Moreover, we believe that planning is the more significant aspect and the aspect which is more demanding of learning. Execution, while not necessarily trivial, is more "mechanical".

It is also the case that planning tends to pass without comment from the student. (We had one subject who preferred to pass through this stage banging his hands on his forehead.) However, we have tried to open this stage up to analysis through the gathering of protocols. These protocols indicate a lot of systematic goings-on which seem to fit under the title of planning.

A plan, in the sense we are using it here, is an outline for action -- the action in this case being proof execution. We believe that the plar students emerge with is a specification of a set of geometric rules that allows one to get from the givens of the problem, through intermediate levels of statements, to the to-be-proven statement. We call such a plan a *proof tree*.

Figure 1 illustrates (a) an example geometry problem, (b) a proof tree, and (c) a proof generated from the tree. In the tree, the goal to prove two angles congruent leads to the subgoal of proving the triangles $\triangle XVZ$ and $\triangle WVY$ congruent. This goal is achieved by the side-angle-side (SAS) postulate. The first side $\overline{VX} \cong \overline{VW}$ is gotten directly from the givens. Since these sides form an isosceles triangle, they also imply $\angle VXZ \cong \angle VWY$, the second part of the SAS congruence pattern. The third part $\overline{XZ} \cong \overline{WY}$ can be gotten from the other given that $\overline{XY} \cong \overline{WZ}$. The proof in Figure 1 is gotten by unpacking various links in the proof tree. It should be noted that some of these links map into multiple

(a)



Given: $\overline{VX} \cong \overline{VW}$, $\overline{XY} \cong \overline{WZ}$
Prove: $\angle XVZ \cong \angle WVY$

(c)

1.	$\overline{VX} \cong \overline{VW}$	given
2.	$\triangle XYW$ is isosceles	definition
3.	$\angle VXZ \cong \angle VWY$	base $\angle$'s of isosceles
4.	$\overline{XY} \cong \overline{WZ}$	given
5.	$XY = WZ$	def. of $\cong$
6.	$YZ = YZ$	symmetric
7.	$XY + YZ = YZ + WZ$	addition
8.	$XZ = XY + YZ$	segment addition
9.	$WY = YZ + WZ$	segment addition
10.	$\overline{XZ} = \overline{WY}$	substitution
11.	$\overline{XZ} \cong \overline{WY}$	def. of $\cong$
12.	$\triangle XVZ \cong \triangle WVY$	SAS
13.	$\angle XVZ \cong \angle WVY$	congruent parts of congruent $\triangle$'s

(b)

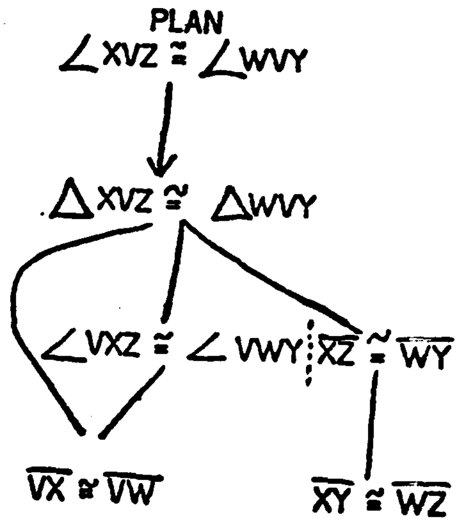


Figure 1

lines of proof. For instance, the link connecting $\overline{XY} \cong \overline{WZ}$ to $\overline{XZ} \cong \overline{WY}$ maps into the 8 lines 4-11 in the proof. This is one of the important reasons why we characterize the proof tree as an *abstract* specification of a proof.

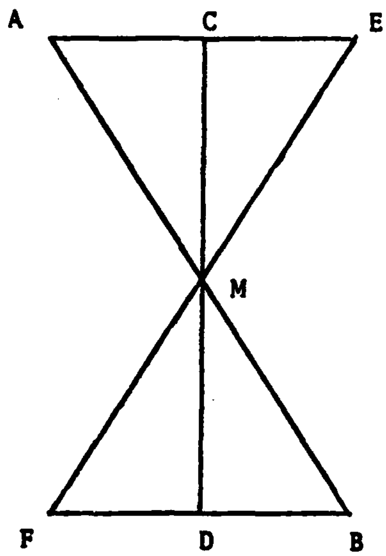
The proof tree is not something that students typically draw out for themselves. Rather it is a knowledge structure in the head. Various remarks of students suggest to us that it is a real knowledge structure not just a product of our theoretical fantasies. For instance, one student described a proof as "an upside down pyramid". For a student a proof tree would be upside-down since the actual proof ends in the to-be-proven statement. However, we display the tree right-side up (to-be-proven statement at the top) because it serves to facilitate theoretical discussion.

As a pedagogical aside, we might say that we think it very important that the central role of such abstract plans be realized in instruction. It is very easy to direct instruction to proof execution and ignore the underlying plan that guides the writing of the proof.

Finding a proof tree is not a trivial problem. The student must either try to search forward from the givens trying to find some set of paths that converge satisfactorily on the to-be-proven statement or he must try to search backward from the to-be-proven statement trying to find some set of dependencies that lead back to the givens. Using unguided forward or backward search, it is easy to become lost. We will argue that students use a mixture of forward and backward search. This mixture, in addition with various search heuristics they acquire, enables students to deal with the search demands of proof problems found in high school geometry texts.

Example: The Simulation

We would like to discuss an example problem derived from Chapter 4 of Jurgensen, Donnelly, Maier, and Rising (1975). This problem is illustrated in Figure 2. It is among the most difficult problems found in that chapter. We would first like to discuss how our ACT simulation performed on this problem. This will serve to illustrate more fully our conception of the planning process in proof generation and how this planning is achieved in a production system. Then we will see how ACT's performance compares with that of a high school subject.



GIVEN: M is the midpoint of $\overline{AB}$ and $\overline{CD}$

PROVE: M is the midpoint of $\overline{EF}$

Figure 2

ACT's search for a proof tree involves simultaneously searching backward from the to-be-proven statement and searching forward from the givens. An attempt is made to try to bring these two searches together. This search process creates a network of logical dependencies. When successful ACT will eventually find in its search some set of logical dependencies that defines a satisfactory proof tree. This proof tree will be embedded within the search network. This larger network we call the *problem net*.

Figure 3 illustrates the problem net at an early state of its development for the problem in Figure 2. The first two reasoning forward productions to apply are

P1: IF X, Y, and Z are on a line
and U, Y, and V are on a line
THEN $\angle XYU \cong \angle ZYU$ because they are vertical angles

P2: IF Y is the midpoint of $\overline{XZ}$
THEN $\overline{XY} \cong \overline{YZ}$ by definition

(Both of these productions and the others in this paper are given in a considerably more informal syntax than what is implemented in the ACT production system. However, it is our judgment that the renditions above are considerably more intelligible and do not omit much that is essential.) The first production, for vertical angles, generates from the diagram in Figure 2 that $\angle AMC \cong \angle BMD$ and that $\angle CME \cong \angle DMF$. This is indicated in Figure 3 from arrows leading from the *vertical angles* reason to the angle congruences. The second production translates the two givens about midpoints into inferences about line congruence.

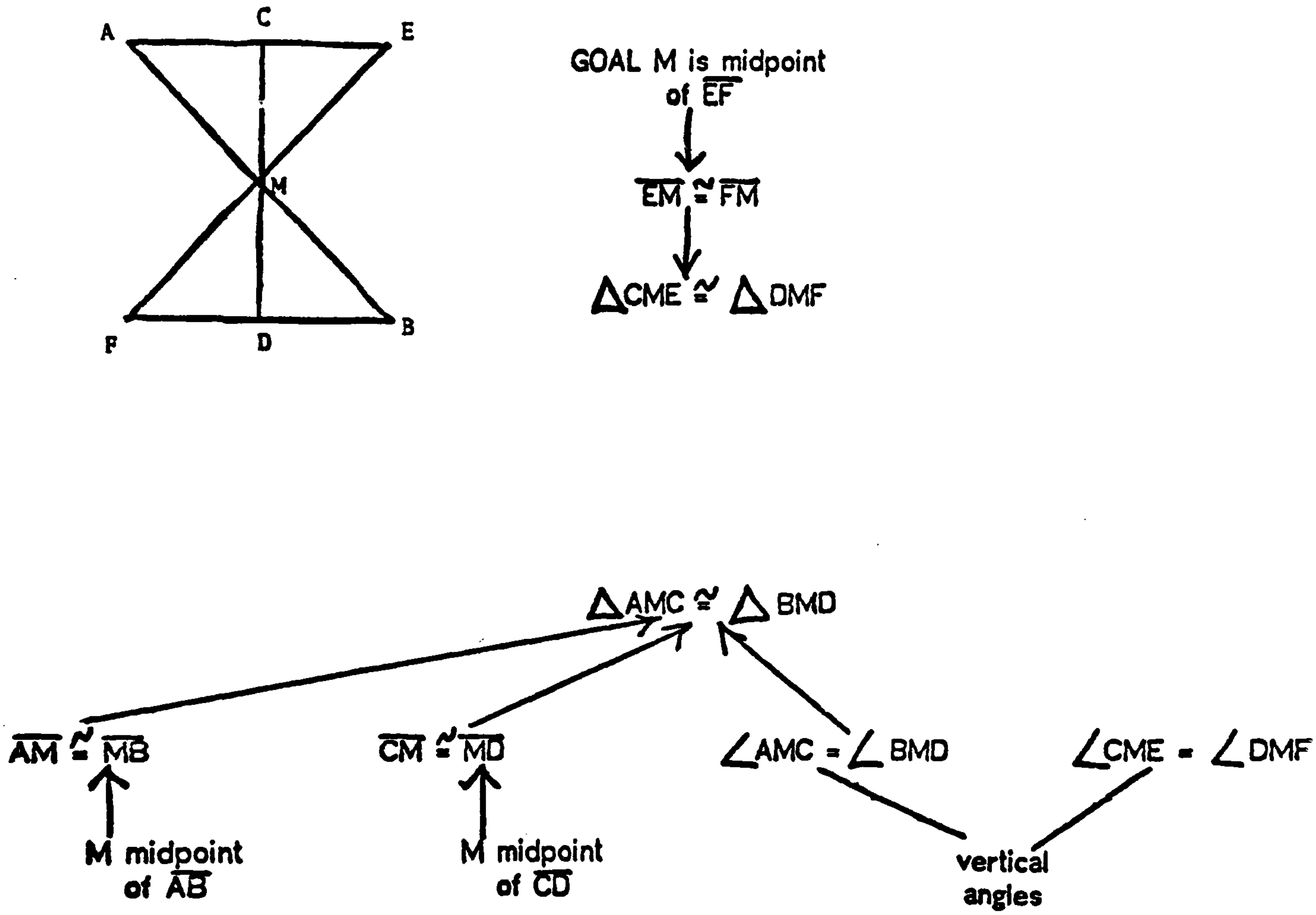
With this information in hand the following working forward production can apply:

P3: IF $\overline{XY} \cong \overline{UV}$
and $\overline{ZY} \cong \overline{WV}$
and $\angle XYZ \cong \angle UVW$
THEN $\triangle XYZ \cong \triangle UVW$ because of SAS

This production embodies the side-angle-side rule (SAS). Applied to the first level of forward inferences in Figure 3 it adds the inference that $\triangle AMC \cong \triangle BMD$. It has been our experience that almost everyone presented with this problem works forward to this particular inference as the first step to solving the problem.

Meanwhile ACT has begun to unwind a plan of backward inferences to achieve the goal. It has

4-A



<u>Figure 3</u>

translated the midpoint goal to a goal of proving the congruence $\overline{EM} \cong \overline{FM}$. This is accomplished by the following production rule:

P4: IF the goal is to prove that Y is the midpoint of $\overline{XZ}$
THEN set as the subgoal to prove $\overline{XY} \cong \overline{YZ}$

This in turn is translated by the following production:

P5: IF the goal is to prove $\overline{XY} \cong \overline{UV}$
and $\overline{XY}$ is part of triangle 1
and $\overline{UV}$ is part of triangle 2
THEN set as the subgoal to prove that triangle 1 is congruent to triangle 2

Matching this production to the diagram, ACT determines that $\triangle$CME contains $\overline{EM}$ an that $\triangle$DMF contains $\overline{FM}$. This leads to the subgoal of proving $\triangle CME \cong \triangle DMF$.

Note that the forward inferences have progressed much more rapidly than the backward inferences. This is because backward inferences, manipulating a single goal are inherently serial whereas the forward inferences can apply in parallel. With respect to the serial-parallel issue it should be noted that the backward and forward search progress in parallel.

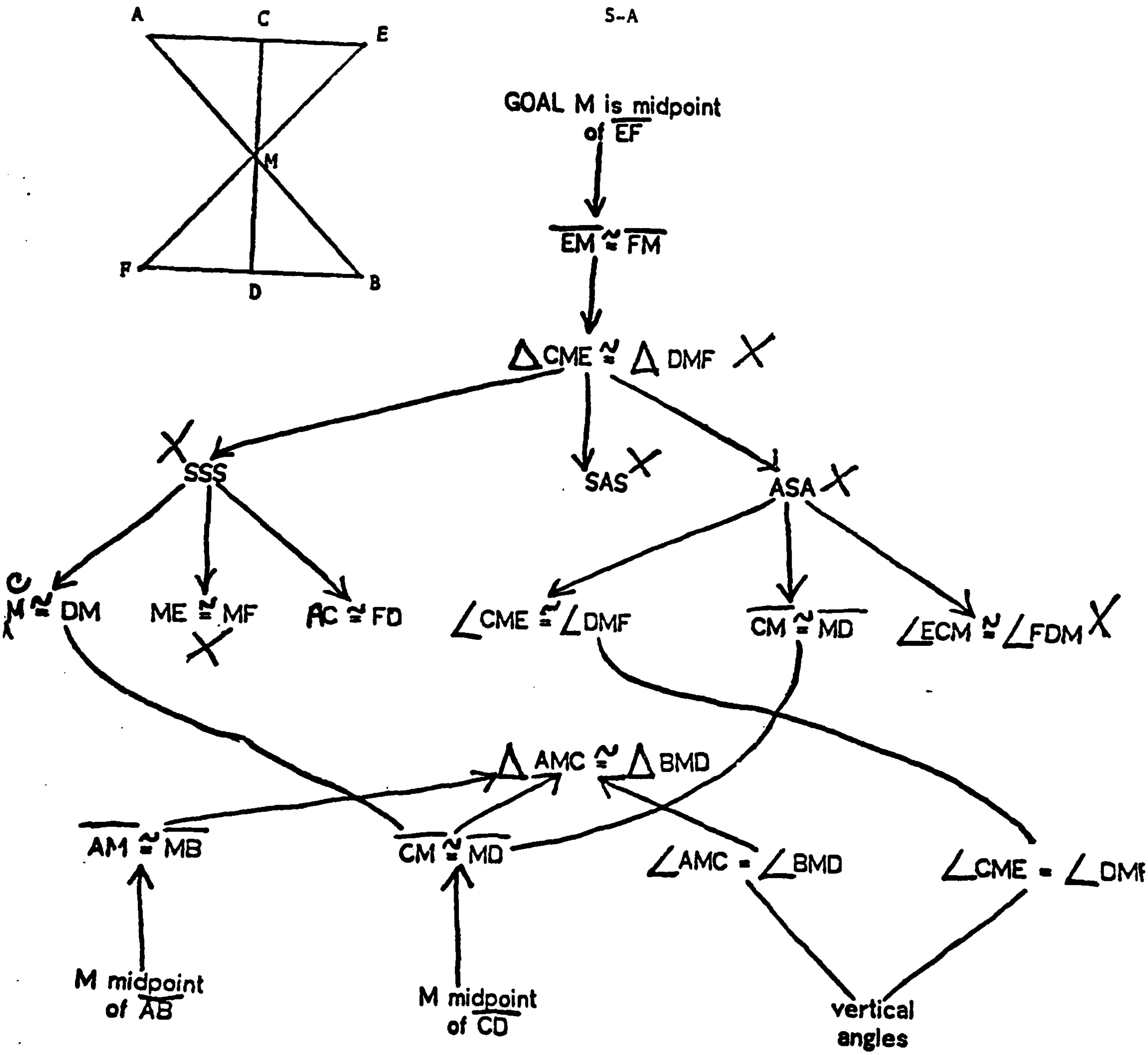
Figure 3 illustrates the limit to the forward inferences that ACT generates. While there are, of course, more forward inferences that could be made, this is the limit to the inferences for which ACT has strong productions available.

Figure 4 illustrates the history of ACT's reasoning backward efforts to establish that $\triangle CME \cong \triangle DMF$. ACT first attempts to achieve this by the side-side-side (SSS) postulate. This subgoal is set by the following production:

P6: IF the goal is to prove that $\triangle XYZ \cong \triangle UVW$
THEN try to use SSS by proving $\overline{XY} \cong \overline{UV}$, $\overline{YZ} \cong \overline{VW}$, and $\overline{ZX} \cong \overline{WU}$

This effort is doomed to failure because the triangle congruence has been set as a subgoal of proving one of the sides congruent. When ACT gets to the goal of establishing $\overline{EM} \cong \overline{FM}$ it recognizes the problem and backs away. Our subject, like ACT, had a certain propensity to plunge into hopeless paths. Presumably one component of learning is to stop setting such hopeless subgoals.

We will skip over ACT's unsuccessful attempt to achieve the triangle congruence by side-angle-side (SAS) and look in detail at its efforts with the angle-side-angle (ASA) postulate. Two



A
C
E
M
F
D
B
S-A
GOAL M is midpoint of EF
EM ≅ FM
△CME ≅ △DMF
SSS
SAS
ASA
CM ≅ DM
ME ≅ MF
AC ≅ FD
∠CME ≅ ∠DMF
CM ≅ MD
∠ECM ≅ ∠FDM
△AMC ≅ △BMD
AM ≅ MB
CM ≅ MD
∠AMC = ∠BMD
∠CME = ∠DMF
M midpoint of AB
M midpoint of CD
vertical angles

Figure 4

of the three pieces required for this -- $\angle CME \cong \angle DMF$ and $\overline{CM} \cong \overline{MD}$ have already been established by forward inferences. This leaves the third piece to be established -- that $\angle ECM \cong \angle FDM$. This can be inferred by supplementary angles from something that is already known -- that $\triangle AMC \cong \triangle BMD$. However, ACT does not have the postulate for making this inference available. This corresponds to a blindness of our subject with respect to using the supplementary rule. Although the opportunity did not arise in this problem because he was following a different path to solution, many times he overlooked opportunities to achieve his goals by the supplementary angle rule.

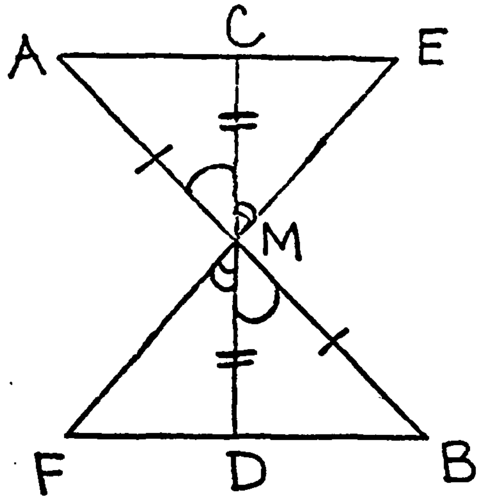
Having failed the three available methods for proving triangle congruence, ACT backed up and found a different pair of triangles, $\triangle AME$ and $\triangle BMF$, whose triangle congruence would establish the higher goal that $\overline{EM} \cong \overline{FM}$. (It turns out that, by failing on the supplementary angle needed to establish $\triangle CME \cong \triangle DMF$ and going on to try $\triangle AME \cong \triangle BMF$, ACT finds the shorter proof.)

Fortuitously, ACT chooses ASA as its first method. The attempt to apply this method is illustrated in Figure 5. One of the angle congruences is obtained by the following working backward rule:

P7: IF the goal is to prove that $\angle XYZ \cong \angle UYW$
and X, Y, and W are on a line
and Z, Y, and W are on a line
THEN the goal can be inferred because of vertical angles

Note that this inference was not made by the forward-reasoning vertical-angle production. This turns out due to a difficulty that the ACT pattern-matcher has in seeing lines define multiple angles. The segments $\overline{AM}$ and $\overline{ME}$ that define $\angle AME$ were already used in extracting the angles $\angle AMC$ and $\angle CME$ for use by the forward reasoning vertical angle postulate. We will discuss later problems human subjects have analogous to this problem.

ACT is also able to get the other parts of the ASA pattern. The side $\overline{AM} \cong \overline{BM}$ has already been gotten by forward inference. The fact the $\angle EAM \cong \angle FBM$ can be inferred from the fact that $\triangle AMC \cong \triangle BMD$ since the angles are corresponding parts of congruent triangles. With this ACT has found its proof tree embedded within the search net. That proof tree is highlighted in Figure 5.


A
C
E
M
F
D
B

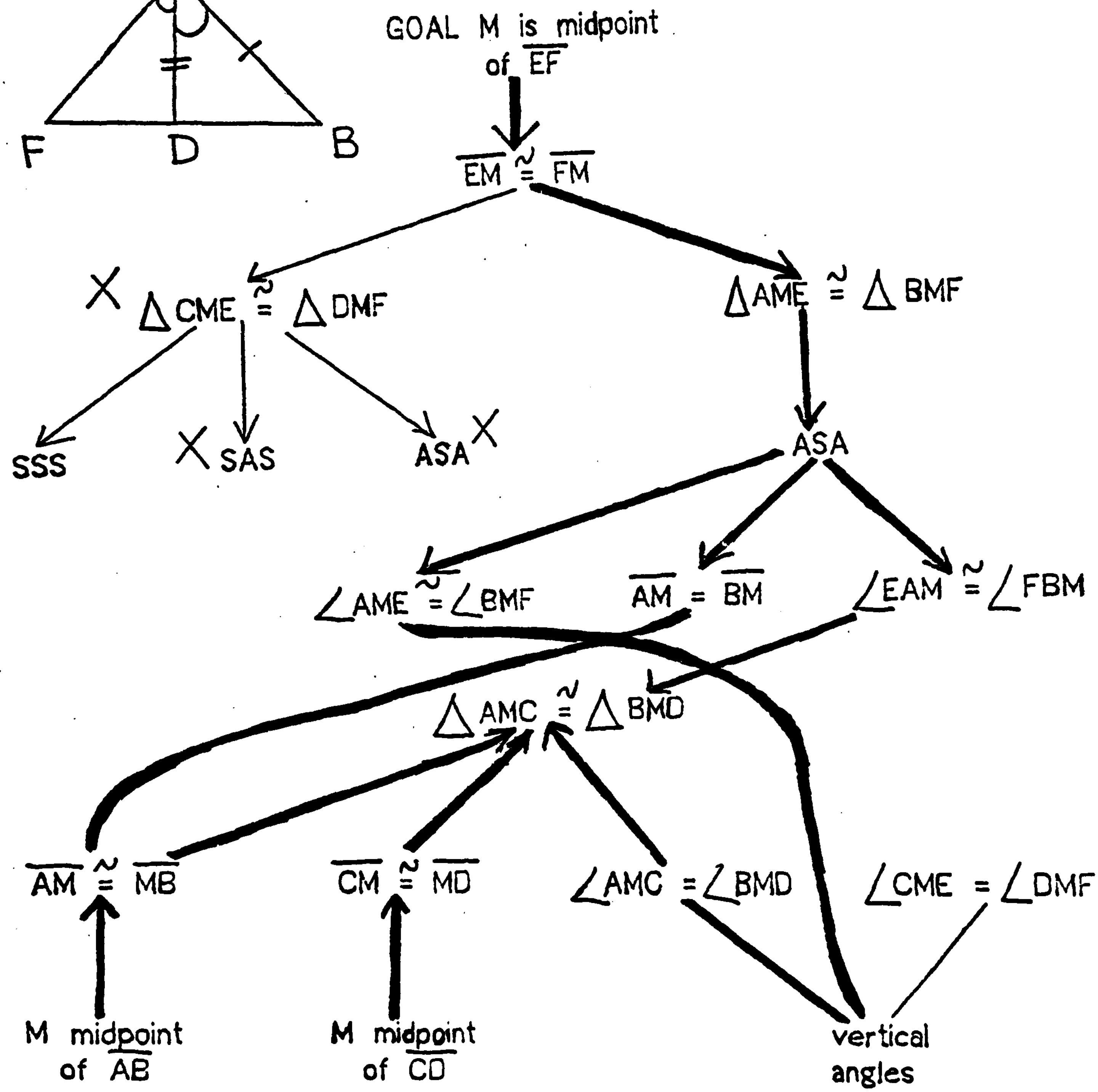

GOAL M is midpoint of $\overline{EF}$
$\overline{EM} \cong \overline{FM}$
X $\triangle CME \cong \triangle DMF$
$\triangle AME \cong \triangle BMF$
SSS
X SAS
ASA X
ASA
$\angle AME \cong \angle BMF$
$\overline{AM} = \overline{BM}$
$\angle EAM \cong \angle FBM$
$\triangle AMC \cong \triangle BMD$
$\overline{AM} \cong \overline{MB}$
$\overline{CM} \cong \overline{MD}$
$\angle AMC = \angle BMD$
$\angle CME = \angle DMF$
M midpoint of $\overline{AB}$
M midpoint of $\overline{CD}$
vertical angles

Figure 5

Comparison of ACT to Subject R

It is of interest to see how ACT's behavior compares to that of a typical student. We have gathered extensive protocols from one subject, R. R took geometry from us in grade 7 as a special enrichment opportunity one year before he would normally take geometry in school. We have a more or less complete record of all his learning and work at geometry through Chapter 4 of Jurgensen, Donnelly, Maier, and Rising. In particular, we have a record of his performance on the critical problem in Figure 2.

Subject R's performance did not correspond to that of ACT in all details. This is to be expected because ACT's choices about what productions to apply have an important probabilistic component to them. However, we can still ask whether ACT and subject R have the same character to their inferences. One way of defining this is whether ACT could have produced R's protocol if the probabilities came out correct. By this criterion ACT is compatible with much of R's protocol.

Like ACT, R began by making the forward inferences necessary to conclude $\triangle AMC \cong \triangle BMD$ and then making this conclusion. Like ACT these inferences were made with little idea for how they would be used. Then like ACT, R began to reason backward from his goal to prove that M was the midpoint of $\overline{EF}$ to the goal of proving triangle congruence. However, unlike ACT he was lucky and chose the triangle $\triangle AME \cong \triangle BMF$ first. Unlike ACT again, but this time unlucky, he first chose SAS as his method for establishing the triangle congruence. He got $\overline{AM} \cong \overline{MB}$ from previous forward inference and the $\angle EAM \cong \angle FBM$ from the fact that $\triangle AMC \cong \triangle BMD$ -- just as ACT obtained this in trying to use ASA. However, he then had to struggle with the goal of proving $\overline{AE} \cong \overline{BF}$. Unlike ACT, subject R is reluctant to back up and he tenaciously tried to find some way of achieving his goal. He was finally told by the instructor to try some other method. Then he turned to ASA. He already had two pieces of the rule by his efforts with SAS and quickly got the third component $\angle AME \cong \angle BMF$ from the fact that they were vertical angles. Note that subject R also failed to make this vertical angle inference in forward mode and only made it in backward mode.

In conclusion, we think that R's behavior is very similar in character to that of ACT. The only major exception is R's reluctance to back up when a particular method is not working out.

Learning

What we have described so far is a general framework in which a student can plan proofs. We believe that much of the basic architecture is a reflection of general methods the student brings to geometry for solving problems. However, the basic architecture is not enough to enable the student to be successful and facile at planning proofs in geometry. The student obviously must learn things specific to geometry. We would like to discuss three classes of learning that are important to geometry. One is learning to represent proof problems in ways that best enable his operators to apply. A second is to get his operators into a form where they will be reliably evoked. (Operators in this discussion correspond to productions that make the forward inferences or the backward steps of reasoning.) Third he must tune his operators so that they will be selected in the appropriate situations and not selected in situations where they will not achieve the goals. In the following sections we will discuss each of these issues.

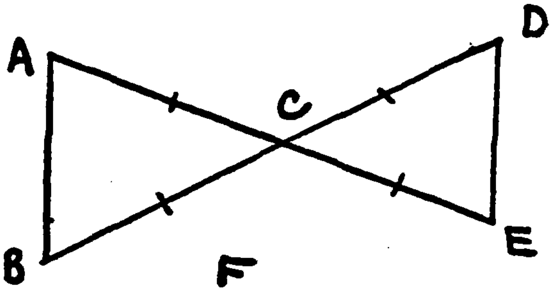
Problem Representation

Geometry introduces to a high school student a number of novel concepts and a rather elaborate set of notations for referring to facts about a problem. To provide an example, students find the concept of included angles difficult and have some problem appreciating the fact that the measure of an angle is independent of the length of the lines that include it. These concepts require the student or ACT to acquire new pattern recognition facilities.

There is another problem students have in geometry that reflects, we think, a fundamental limitation on human pattern recognition. Students have difficulty perceiving the same object participating in multiple patterns. Figure 6 illustrates three instances of this. The proof in part (a) requires that students recognize the vertical angles. However, students have difficulty in doing this and at the same time seeing these as angles of a triangle. The proof in part (b) requires that students see the base angles of the super triangle as also angles of the smaller triangles. Students have difficulty in recognizing such shared angles. The problem in (c) requires that the student see the segment $\overline{AC}$ as a component of two triangles. Subject R. on one of his early encounters with the kind of problem that is in (c), was so bothered by this that he invented a construction which sliced the line

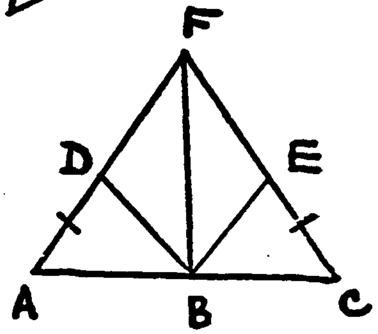
Problem 2 in Data Representation: The same data filling multiple roles

(a)



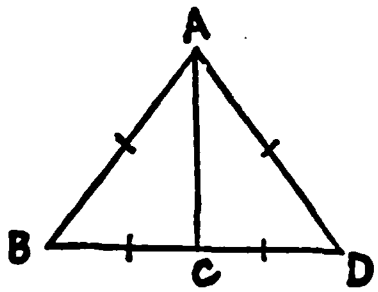
Given: $\overline{AC} \cong \overline{CE}$
$\overline{BC} \cong \overline{CD}$
Prove: $\triangle ABC \cong \triangle EDC$

(b)



Given: $\triangle AFB \cong \triangle CFB$
$\overline{AD} \cong \overline{CE}$
Prove: $\triangle AFB \cong \triangle CEB$

(c)



Given: $\overline{AB} \cong \overline{AD}$
$\overline{BC} \cong \overline{CD}$
Prove: $\triangle ABC \cong \triangle ACD$

Figure 6

$\overline{AC}$ in two.

There are two ways that students appear to deal with these problems. One is to learn larger patterns and associate the relevant information with these patterns. So, for instance, the subject R learned to respond to the pattern in (a) with the inference that the two triangles contained congruent vertical angles. The other solution is to acquire metarules such as looking to see if an angle appears in other triangles. This basically forces the student to reorient his pattern-matching.

We have not worked to apply the ACT theory to acquisition of the kind of pattern recognition skills required to deal with these problems. We mention them here mainly because they prove to be a significant part of the learning problem. We have applied ACT to the other two aspects of learning which are also quite significant learning problems for humans.

Proceduralization of Operators

When students read a definition, postulate, or theorem, it seems unreasonable to suppose that they immediately convert it into a procedural form such as the productions presented in the discussion of Figures 3 through 5. One reason that it is unreasonable is that the same fact of geometry can give rise to a great many possible productions reflecting various ways that the information can be used. For instance, consider the textbook definition of supplementary angles:

"Supplementary angles are two angles whose measures have sum 180."

Below are productions that embody just some of the ways in which this knowledge can be used. These productions differ in terms of whether one is reasoning forward or backward, what the current goal is, and what is known.

P8: IF $m\angle A + m\angle B = 180°$
THEN $\angle A$ and $\angle B$ are supplementary

P9: IF the goal is to prove $\angle A$ and $\angle B$ are supplementary
THEN set as a subgoal to prove $\angle A + \angle B = 180°$

P10: IF $\angle A$ and $\angle B$ are supplementary
THEN $m\angle A + m\angle B = 180°$

P11: IF $\angle A$ and $\angle B$ are supplementary
and $m\angle A = X$
THEN $m\angle B = 180° - X$

P12: IF $\angle A$ and $\angle B$ are supplementary
and the goal is to find $m\angle A$
THEN set as a subgoal to find $m\angle B$

P13: IF the goal is to show $\angle A \cong \angle B$
and $\angle A$ is supplementary to $\angle C$
and $\angle B$ is supplementary to $\angle D$
THEN set as a subgoal to prove $\angle C = \angle D$

A basic point is that the definition of supplementary angles is fundamentally declarative in the sense that it can be used in multiple ways and does not contain a commitment to how it will be used. It is unreasonable to suppose that, in encoding the definition, the system anticipates all the uses to which it might be used and creates a procedural structure for each.

A related difficulty has to do with encoding control information into working-backward productions. The actual implementations of a working-backward production requires rather intricate knowledge and use of goal settings. It seems unreasonable to propose that the learner has conscious access to this kind of information in the way necessary for direct encoding of productions.

Rather than assuming students directly encode this textbook information into procedures we assume that they first encode this information declaratively. In the ACT system encoding information declaratively amounts to growing new semantic network structure to encode the information. We suppose general *interpretative* procedures then use this information according to the features of the particular circumstance. As we will describe, when declarative knowledge is used multiple times in a particular way, automatic learning processes in the ACT theory will begin to create new procedures that directly apply the knowledge without the interpretative step. We refer to this kind of learning as *procedural compilation*.

In individual subjects we see a gradual shift in performance which we would like to put into correspondence with this compilation from the interpretative application of declarative knowledge to direct application of procedures. After reading, say a particular postulate, students' applications of that postulate is both slow and halting. Students will often recite to themselves the postulate before trying to apply it -- or even go back and reread it. It seems that they need to activate the declarative

representation in their working memory so that interpretative procedures can apply to the data of this representation. They typically match separately fragments of the postulate to the problem. We will see that such fragmentary application is typical of a general knowledge interpreter applying to a declarative representation. With repeated use, however, application of the postulate smooths out. It is no longer explicitly recalled and it is no longer possible as observer or subject to discriminate separate steps in the application of the procedure. It certainly has the appearance of the postulate being embodied in separate pattern recognition productions such as those described with respect to Figures 3 through 5.

Knowledge Schemas

We have found a schema-like representation to be very useful for structuring the initial declarative encoding of a geometry fact. Table 1 illustrates a schema encoding for the SAS postulate which is stated in the text as

> If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.

The diagram in Figure 7 accompanied this statement. The postulate schema in Table 1 is divided into background, hypothesis, conclusion, and comment. The hypothesis and conclusion reflect the if-then structure of the condition at which our subject was fairly facile at extracting. The background information amounts to a description of the diagram and contains the constraints which allow the variables (sides and angles) to be properly bound. The comment contains additional information relevant to its use. Here we have the name of the postulate which prescribes what the student should write as a reason.

In reading such a postulate subject R would typically read through at a slow but constant rate and then go to the diagram trying to relate it to the statement of the postulate. More time would be spent looking at the diagram and relating it to the postulate statement than anything else. We take this to indicate time spent extracting the background information which is not very saliently presented for a particular problem. Students are not always successful at extracting the relevant background information. For instance, subject R failed to appreciate what was meant by "included angles" (hence the question marks around these clauses in the background in Table 1). It was only sometime

Side-Angle-Side Postulate: If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.

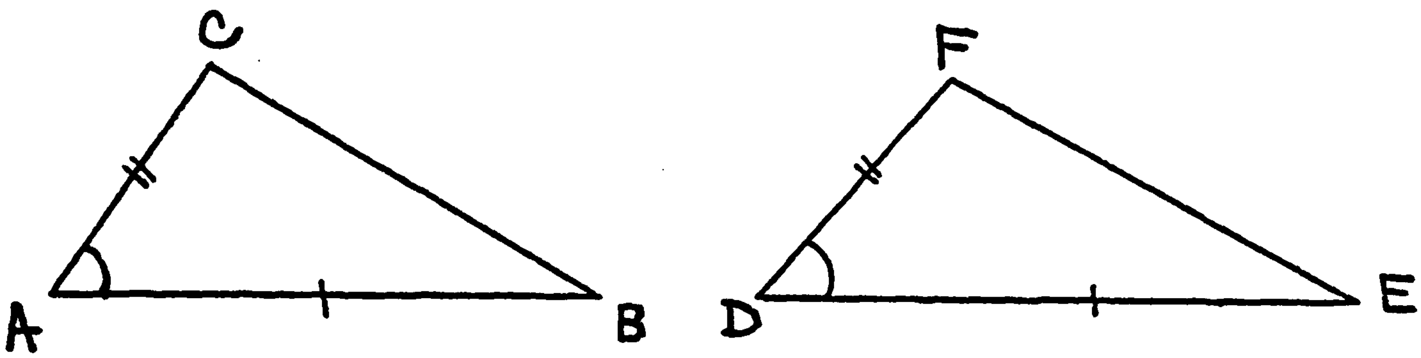


Figure 7

later, after direct intervention by the experimenter, that R got this right.

We regard the knowledge structure in Table 1 to be schema-like in that it is a unit organized into parts according to "slots" like background, hypothesis, conclusion, and comment. The knowledge structure is declarative in that it can be used in multiple ways by interpetative procedures. For instance, the following production would be evoked to apply that knowledge in a working-backwards manner:

P14: IF the goal is to prove a statement
and there is a schema that has this statement as conclusion
THEN set as subgoals to match the background of the schema
and after that to prove the hypothesis of the schema

If the schema is in working memory and its conclusion matches the current goal, this production will start up the application of the schema to the current problem. First the background is matched to bind the variables and then the hypotheses are checked.

The same schema can be evoked multiple ways in a working forward mode. For instance, the following production would serve to evoke the schema in this manner:

P15: IF a particular statement is known to be true
and there is a schema that includes this statement in its hypothesis
THEN set as subgoals to match the background of the schema
and then to match the remaining hypotheses of the schema
and if they match, to add the conclusion of the schema

This production is evoked when only part of the hypothesis is satisfied. Because there can be any number of statements in a hypothesis, it is not possible to have general interpretive productions that match all the statements of any hypothesis. Rather it is necessary to evoke working forward when it is found that a subset of the conditions match and then to check if the remaining statements match. This is one instance of many that illustrates the need for piecemeal application when knowledge is used interpretively. Before the rest of the hypotheses can be checked the background must be matched to bind variables. If the hypotheses do match, the conclusion is added as an inference.

Note that whether the knowledge is used in reasoning forward or reasoning backward, the background must be matched first. In reasoning forward, the hypotheses serve as a test of the applicability of the schema and the conclusion is added. In reasoning backwards, the conclusion

SAS Schema

Background

s1 is a side of △ XYZ
s2 is a side of △ XYZ
A1 is an angle of △ XYZ
?A1 is included by s1 and s2?
s3 is a side of △ UVW
s4 is a side of △ UVW
A2 is an angle of △ UVW
?A2 is included by s3 and s4?

Hypothesis

s1 is congruent to s3
s2 is congruent to s4
A1 is congruent to A2

Conclusion

△ XYZ is congruent to △ UVW

Comment

This is the side-angle-side postulate

Table 1

serves as the test and the hypotheses are added as subgoals. However, in either mode the background serves as a precondition that must be satisfied.

Knowledge Compilation

To appreciate how learning switches the student from the initial piecemeal interpretive application to direct, unitary procedures it would be useful to sketch out a few more of the productions that are used in the initial interpretive application. Let us consider some of the productions involved in working backwards. After production P14, which starts things, the next production to apply would be

P16: IF the goal is to match a set of statements
THEN match the first statement in the set

Production P14 had set the subgoal of matching the statements in the background. This production above starts that process going by focusing on the first statement in the background. This production is followed by a production which iterates through the statements of the background.

P17: IF the goal is to match a statement in a set
and the problem contains a match to the statement
THEN go on to match the next statement in the set

(Actually, there is a call to a subroutine of productions which execute the matches to each statement. See Neves & Anderson, in preparation). After all statements in the background have been matched, the following production sets the goal to prove the hypotheses:

P18: IF the goal is to match a set of statements
and the last statement in that set has been matched
THEN go on to the goal that follows

Composition

There are two major processes in knowledge compilation -- *composition* and *instantiation*. When a series of productions apply in a fixed order, composition will create a new production that accomplishes the effect of the sequence in a single step (see Neves & Anderson). Composition, operating on the sequence of P14, P16, and P17, applied to the SAS schema, would put forth the production

P19: IF the goal is to prove a statement
and there is a schema that has this statement as conclusion
and the schema has a statement as the first member of its background
and the problem contains a match to the statement
THEN set as subgoals to match the background

and within this subgoal to match the next statement of the background
and after that to prove the hypotheses of the schema

This production only applies in the circumstance that the sequence P14, P16, and P17 applied and has the same effect in terms of changes to the data base. The details underlying composition are discussed in Neves and Anderson, but the gist of the process is easy to describe. The composed production collects in its condition all those clauses from the individual productions' conditions except those that are the product of the actions of earlier productions in the sequence. As an example of this exception P16 has in its condition that the goal is to match the set of statements. Since this goal was set by P14, earlier in the sequence, it is not mentioned in the condition of the composed production P19. Thus, the condition is a test of whether the circumstances are right for the full sequence of productions to execute. The action of the composed production collects all actions of the individual productions except those involved in setting transitory goals that are finished with by the end of the sequence. As an example of this exception, P16 sets the subgoal of matching the first statement of the background but P17 meets this subgoal. Therefore, the subgoal is not mentioned in the action of the composed production P19.

This composition process can apply to the product of earlier compositions. Although there is nothing special about compositions of three, consider what the resulting production would be like if P19 were composed with two successive iterative applications of P17:

P20: IF the goal is to prove a statement
and there is a schema that has this statement as conclusion
and the schema has a statement as the first member of the background
and the problem contains a match to this statement
and the schema has another statement as the next member of its background
and the problem contains a match to this statement
and the schema has another statement as the next member of its background
and the problem contains a match to this statement
THEN set as subgoals to match the background
and within this the next statement of the background
and after that to prove the hypotheses of the schema

It should be noted that such productions are not really specific to the SAS schema. Indeed, productions such as P19 and P20 might have been formed from compositions derived from the productions applying to other, earlier schemata. If so, these composed productions would be ready

to apply to the current schema. Thus, there can be some general transfer of practice through composition. However, there is a clear limit on how large such composed productions can become. As they get larger they require more information in the schema be retrieved from long term memory and held active in working memory. Limits on the capacity of working memory imply limits on the size of the general, interpretive conditions that can successfully match.

Instantiation

Instantiation is a process that builds specialized versions of productions by eliminating retireval of information from long term memory. Rather the information that would have been retrieved from long term memory is built into the specialized version of the production. To illustrate the process of instantiation, consider its application to the production P20. This statement contains in its condition four clauses that require retrieval of information from long term memory:

1. There is a statement that has this conclusion.

2. The schema has a statement as the first member of its background.

3. The schema has another statement as the next member of its background.

4. The schema has another statement as the next member of its background.

Applied to the SAS schema these statements match the following information:

1. The SAS schema has as its conclusion "$\Delta XYZ \cong \Delta UVW$".

2. The first statement of its background is "S1 is a side of ΔXYZ".

3. The next statement of its background is "S2 is a side of ΔXYZ".

4. The next statement of its background is "A1 is an angle of ΔXYZ".

What is accomplished by matching these statements in P20 is to identify the SAS schema, its conclusion, and the first three statements of its background. A specialized production can be built which contains this information and does not require the long term memory retrievals:

P21: IF the goal is to prove that $\Delta XYZ \cong \Delta UVW$
and S1 is a side of ΔXYZ
and S2 is a side of ΔXYZ

and A1 is an angle of ΔXYZ
THEN set as subgoals to match the background of the SAS schema
and within this to match the next statement in the schema
and after that to prove the hypothesis of the schema

This production is now specialized to the SAS schema and does not require any long term memory retrieval. Rather, built into its condition are the patterns retrieved from long term memory.

The effect of this instantiation process is to enable larger composed productions to apply because the instantiated productions are not limited by the need to retrieve information into working memory. This in turn allows still larger compositions to be formed. The eventual product of the composition process applied to the top down evocation of the SAS schema, initially via productions P14, P16, P17, and P18 would be:

P22: IF the goal is to prove that ΔXYZ is congruent to ΔUVW
and S1 is a side of ΔXYZ
and S2 is a side of ΔXYZ
and A1 is an angle of ΔXYZ
and A1 is included by S1 and S2
and S3 is a side of ΔUVW
and S4 is a side of ΔUVW
and A2 is an angle of ΔUVW
and A2 is included by S3 and S4
THEN set as subgoals to prove
S1 is congruent to S3
S2 is congruent to S4
A1 is congruent to A2

This production serves to apply the SAS postulate in working backward mode. When the knowledge reaches this state it has been completely proceduralized.

As we will discuss in later portions of the paper, composition need not stop when the postulate has been completely incorporated into a single production. It can continue to merge productions to compress even longer sequences of actions into a single production. For instance, consider what would happen should production P22 compose with later productions that attempted to prove the hypothesis parts. Suppose, furthermore, that the first two parts of the hypothesis could be established directly since they were already true. The composition process would produce the following working backward production:

P23:. IF the goal is to prove that ΔXYZ is congruent to ΔUVW
and S1 is a side of ΔXYZ

and S2 is a side of ΔXYZ
and A1 is an angle of ΔXYZ
and A1 is included by S1 and S2
and S3 is a side of ΔUVW
and S4 is a side of ΔUVW
and A2 is an angle of ΔUVW
and A2 is included by S3 and S4
and S1 is congruent to S3
and S2 is congruent to S4
THEN set as a subgoal to prove that A1 is congruent to A2

This production checks that two sides of the triangles are congruent and sets the goal to prove that the included angle is congruent. P23 is obviously much more discriminant in its application than P22 and is therefore much more likely to lead to success.

Evidence for Composition and Instantiation

So far we have offered two lines of argument that there are these processes of composition and instantiation. One is that it creates a sensible connection between declarative knowledge and procedural knowledge. That is, knowledge starts out in a declarative form so that it can be used in multiple ways. However, if the knowledge is repeatedly used in the same way, efficient procedures will be created to apply the knowledge in that way. The second argument for these processes is that they are consistent with the gross qualitative features of the way application of knowledge smooths out and speeds up. That is, with practice explicit verbal recall of the geometry statements drop out and the piecemeal application becomes more unitary.

The idea is a natural one, that skill develops by collapsing together multiple steps in one. Lewis (1976), who introduced composition applied to productions, traces the general concept back to Book (1908). However, there is more than intuitive appeal and general plausibility going for this learning mechanism. It is capable of accounting for a number of important facts about skill development. One feature of the knowledge compilation is that procedures can develop to apply the knowledge in one manner without corresponding procedures developing to apply the knowledge in other ways. It is somewhat notorious that people's ability to use knowledge can be specific to how the knowledge is evoked. For instance, Greeno, Magone, Goldberry, and Mokwa (preliminary draft) find that students who have a fair facility at proof generation make gross errors at proof checking, a skill which they have not practiced.

Another factor about skill development is that there is a characteristic speed-up of the skill with practice at a task. There appear to be power functions (with exponents between 0 and -1) relating time to perform a task to amount of practice. These functions are discussed at length in Neves and Anderson where it is shown that ACT's knowledge composition processes can account for the qualitative character of these practice functions.

ACT's knowledge compilation can also account for many of the phenomenon associated with the movement to automaticity with practice (again as discussed in detail in Neves and Anderson). For instance, there is the loss of awareness of executing the skill. This follows from the fact that knowledge is no longer being retrieved into working memory as it is being used. Thus, there is no information that can be inspected to achieve conscious monitoring of the skill acquisition. A number of the more behavioral indicants associated with automaticity (e.g., Schneider & Shiffrin, 1977) can also be explained. Schneider and Shiffrin find no effect of number of elements being held in memory. This can be explained since the instantiation process does away with the need for memory retrieval. Another feature of the Shiffrin and Schneider analysis is that the effect of number of display alternatives diminishes with practice. This is also expected as a consequence of specialization in knowledge compilation. In the limit instantiation and composition will build special productions to deal with each display alternative and eliminate the need for display scanning.

Another feature of composition is that it will produce the Einstellung phenomenon (Luchins, 1942). The connection between composition and Einstellung is discussed in detail in Lewis and in Neves and Anderson. Basically, composition skips the processing over intermediate states where more appropriate paths in a search problem can be noticed. As a consequence a rigidity will appear after certain types of practice, in that the system misses solutions that do not conform to the established pattern. We will discuss one example of this in the next section on search.

In summary then, we feel that there is much evidence converging on the twin processes of composition and instantiation. The pair play the key role in converting declarative knowledge into a procedural form.

Optimizing Proof Search

Having operators proceduralized is not enough to guarantee successful proof generation. There is still a potentially very large search net of forward and backward inferences. Finding the proof tree in this net would often be infeasible without some search heuristics that cause the system to try the right inferences first.

In truth, we only saw subject R having very modest success in discovering such heuristics. Therefore, our observations of R do not provide a strong basis for the assertion that acquisition of such heuristics is an important part of learning in geometry. Rather, our beliefs on this matter come from comparing our performance on proof problems with that of subject R. While the title "expert" is a little overblown in our case, we have something of a novice-expert contrast here. Subject R barely managed to get his knowledge beyond the initial proceduralization and often made choices in search that seem transparently wrong to us. Presumably, our more tuned judgment reflects the acquisition of appropriate heuristics with experience. Therefore, in discussing particular heuristics we will be drawing both on those rare instances of heuristic learning identifiable in R and our own intuitions about the kinds of heuristics we use.

A heuristic in this discussion amounts to adding some discriminative conditions to a production to restrict its applicability. For instance, production P23 differs from P22 by the addition of tests for two out of three of the conditions of SAS. While satisfying these conditions does not guarantee that SAS will be satisfied, it does make it more likely. This is the nature of a heuristic -- to select operators on the basis of tests that suggest higher-than-average probability of success.

It is interesting to note that novices do not deal with proofs by plunging into endless search. They are very restrictive in what paths they attempt and are quite unwilling to consider all the paths that are legally possible. The problem is, of course, that the paths they select are often non-optimal or just plain dead-ends. Thus, at a general level, expertise does not develop by becoming more restrictive in search, rather it develops by becoming more appropriately restrictive.

There are four ways that we have been able to discover by which subjects can learn to make better

choices in searching for a proof tree. One is by analogy to prior problems -- using with the current problem methods that succeeded in similar past problems. The second, related technique is to generalize from specific problems operators that capture what the solutions to these specific problems have in common. The third is a discrimination process by which restrictions are added to the applicability of more general operators. These restrictions are derived from a comparison of where the general operators succeeded and failed. The fourth process is a composition process by which sequences of operators become collapsed into single operators that apply in more restrictive situations. We will discuss each of these methods of learning search heuristics in turn.

Learning by Analogy

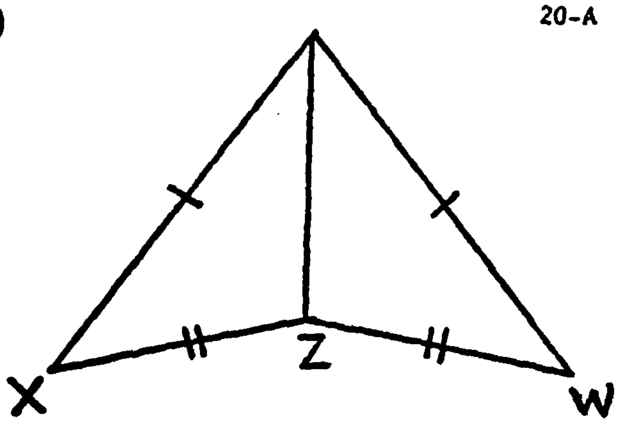
The process of using analogy to past problems can, in some ways, be characterized as a degenerate learning process. Figure 8 illustrates an early opportunity for analogy in the chapter on triangles. The student has just seen a solution to the problem in part (a) and then is presented with the problem in part (b). Our subject R noticed the similarity between the two problems went back to the first and almost copied over the solution.

Analogy of this sort is an interesting kind of learning in that it amounts to learning very specific operators. For example, for the problem in part (a) we would have a schema that described the specific problem and its solution:

PROBLEM SCHEMA
Background
There is a triangle $\triangle XYZ$
There is a triangle $\triangle WYZ$
Givens
$\overline{XY} \cong \overline{WY}$
$\overline{XZ} \cong \overline{WZ}$
Goal
$\triangle XYZ \cong \triangle WYZ$
Method
$\overline{YZ} \cong \overline{YZ}$ by reflexivity
$\triangle XYZ \cong \triangle WYZ$ by SSS
Comment
This problem appears at the beginning of Section 4.2

To account for the effectiveness of analogy we must assume that the student has a facility to partially match the background and givens of one problem to the background and given of another problem.

(a)

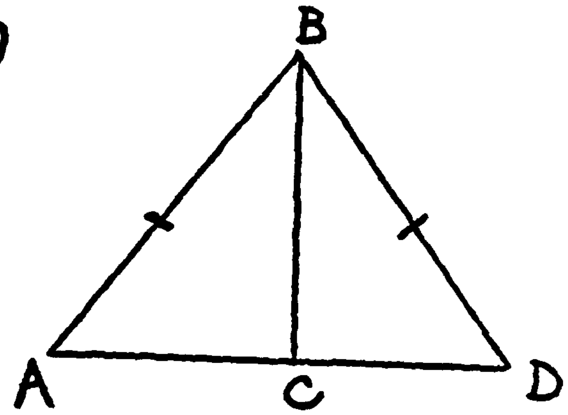


Given: $\overline{XY} \cong \overline{WY}$

$\overline{XZ} \cong \overline{WZ}$

Prove: $\triangle XYZ \cong \triangle WYZ$

(b)



Given: $\overline{AB} \cong \overline{DB}$

$\overline{BC}$ bisects $\overline{AD}$

Prove: $\triangle ABC \cong \triangle DBC$

<u>Figure 8</u>

This is because there is not a perfect match between the two problems. We have recently developed such partial matching facilities for the ACT theory.

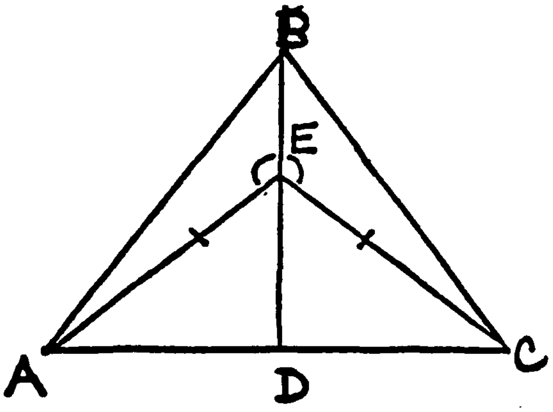
One problem with analogy to specific problems is that it appears to be effective only in the short run because students' memory for specific problems tends to be short-lived. All examples we have of analogy in R's protocols come within the same section of a chapter. We have no examples of problems in one section reminding R of problems in an early section. Therefore, it seems that pure analogy tends to produce no permanent benefits.

A second problem with pure analogy is that it is superficial. Any point of similarity between two problems increases the partial match. It is no accident that the two pairs of triangles in Figure 8 are oriented in the same direction, although this is completely irrelevant for the success of the analogy.

In ACT analogy depends on partial matching processes which are quite "syntactic" in character. That is, the partial match process just counts up the degree of overlap in the problem description without really evaluating whether the overlaps are essential to the appropriateness of the solution or not. In our own selves we note a tendency to respond to overlap between problems in this same superficial way. Consider the three problems in Figure 9. At a deep level the first two problems are really quite similar. Larger triangles contain smaller triangles. To prove the containing triangles congruent it is first necessary to prove the contained triangles congruent. The contained triangles in the two problems are congruent in basically the same way and they overlap with the containing triangles in basically the same way. However, on first glance the two problems seem quite different. In contrast, on first glance, the two problems in parts (a) and (c) of Figure 9 appear to have much in common. Now it is true that upon careful inspection we can determine that the first pair provides a more useful analogy than the second pair. However, it seems that analogy in problem solving of this sort is to serve a *noticing* function. Similar problems spontaneously come to mind as possible models for solutions. If the superficial similarity between problems (a) and (b) is not sufficient for the analogy to be noticed there will never be the opportunity for careful inspection to realize how good the deep correspondence is.

There is one very nice illustration of the problem with the superficiality of analogy in the protocol of

(a)

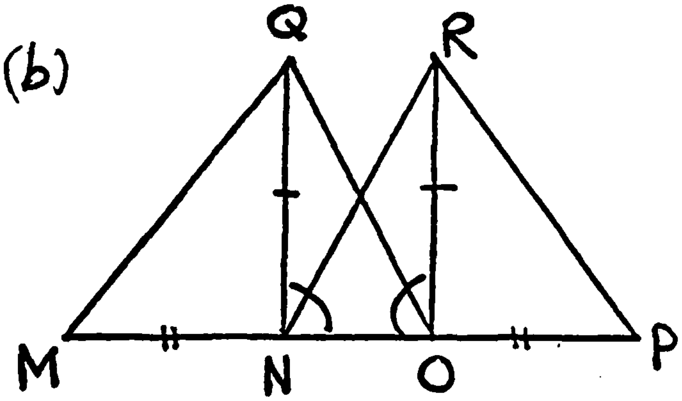


Given: $\overline{AE} \cong \overline{EC}$

$\angle BEA \cong \angle CEA$

Prove: $\triangle ABD \cong \triangle CBD$

(b)



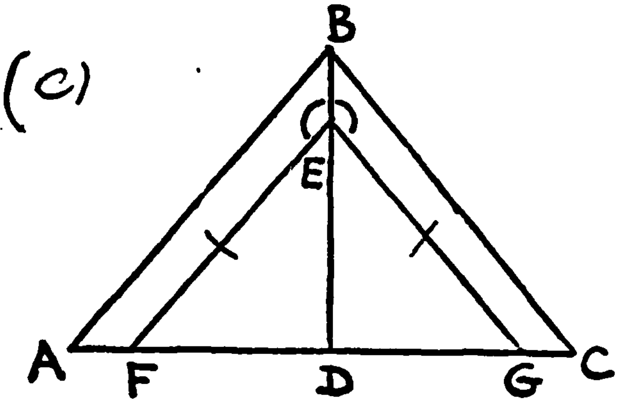
Given: $\overline{QN} \cong \overline{OR}$

$\angle QON \cong \angle RON$

$\overline{MN} \cong \overline{OP}$

Prove: $\triangle MQO \cong \triangle PRN$

(c)



Given: $\overline{FE} \cong \overline{GC}$

$\angle BEF \cong \angle CEG$

$\overline{AB} \parallel \overline{FE}$

$\overline{BC} \parallel \overline{EG}$

Prove: $\triangle ABD \cong \triangle CBD$

Figure 9

R. This concerns a pair of problems that come in the first chapter. Figure 10 illustrates the two problems. Part (a) illustrates the initial problem R studied along with an outline of the proof. Later in the section R came across problem (b) and immediately noticed the analogy. He tried to use the first proof as a model for how the second should be structured. Analogous to the line RO = NY he wrote down the line AB > CD. Then analogous to the second line ON = ON he wrote down BC > BC! His semantic sensitivities caught this before he went on and he abandoned the attempt to use the analogy.

Generalization

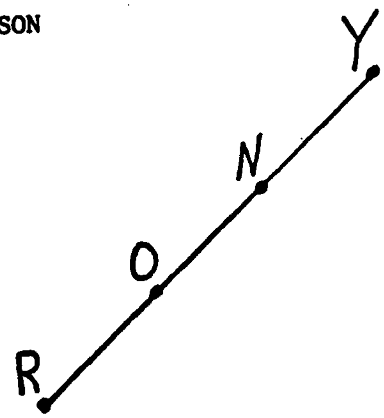
We have characterized solving problems by analogy as superficial. Part of what is superficial about the approach is that the analogy is based only on the statement of the problems not on the structures of their solution. Analogy, in the sense discussed, cannot use the structure of the solution, because the proof for the second problem is not available yet. Analogy is being used in service of finding the second proof.

Generalization, on the other hand, is based on a comparison between two problems and their solutions. By using the structure of the solution it is possible to select out the relevant aspects of the problem statement. A rule is formulated by the generalization process which tries to formulate what the two problems and their solutions have in common. That rule can then be used should similar problems appear. For instance, consider the first two problems in Figure 9. The generalization process applied to these two examples would encode what they have in common by the following schema:

GENERALIZED SCHEMA:

Background
$\triangle XYZ$ contains $\triangle SYZ$
$\triangle UVW$ contains $\triangle TVW$
Givens
$\overline{SY} \cong \overline{TV}$
$\angle YSZ \cong \angle VTW$
Goal
$\triangle XYZ \cong \triangle UVW$
Method
$\triangle SYZ \cong \triangle TVW$ by SAS
$\overline{YZ} \cong \overline{VW}$ by corresponding parts

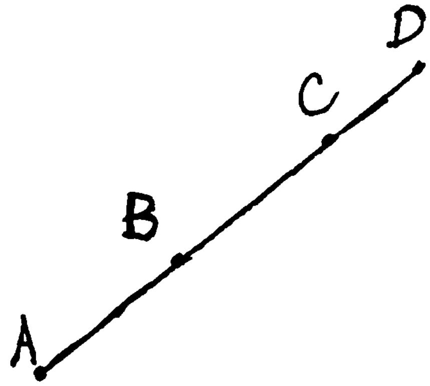
(a)



GIVEN: RO = NY
PROVE: RN = OY

RO = NY
ON = ON
RO + ON = ON + NY
RO + ON = RN
ON + NY = OY
RN = OY

(b)



GIVEN: AB > CD
PROVE: AC > BD

AB > CD
BC > BC

! ! !

Figure 10

$\angle XYZ \cong \angle UVW$ by corresponding parts
$\triangle XYZ \cong \triangle UVW$ by SAS

In our opinion, these generalizations are based on the same partial-matching process that underlies analogy. However, the partial-matching occurs between solved problems not just between problem statements. Because the product of the partial match is a fairly general problem description, it is likely to apply to many problems. Thus it is likely to be strengthened and become a permanent part of the student's repertoire for searching for proofs. This contrasts to the specific examples that serve as the basis for analogy. These specific examples are likely to be forgotten.

We believe, and there is some empirical evidence (Elio & Anderson, in preparation), that it is important to have in close proximity the examples that gave rise to the generalizations. This is predicted by the mechanisms which underlie the generalization processes in ACT. Later, we will expand on the pedagogical implications of the possible importance of contiguity to generalization.

In contrast to analogy, there is rather scant evidence for the importance of generalizations in the proof seeking of R. In part at least, this is due to structure of the exercises which provide little opportunity for generalization. We believe that such generalizations play a moderately important role in our own proof seeking, but we have no systematic evidence about the origin of our own generalizations.

We have been able to identify only two moderately clear cases of generalization in R's protocols. One has to do with problems of the variety illustrated in part (a) of Figure 10. Many variations on this problem appeared in the early part of the text and R came to recognize this general type of problem when it appeared later. For instance, it appears as part of the solution to problem (b) in Figure 9. The other example has to do with the use of the hypotenuse-leg theorem for right angle triangles. After some examples R formulated the generalized rule that he should use this theorem if he was presented with two right angle triangles whose hypotenuses were given as congruent.

Discrimination

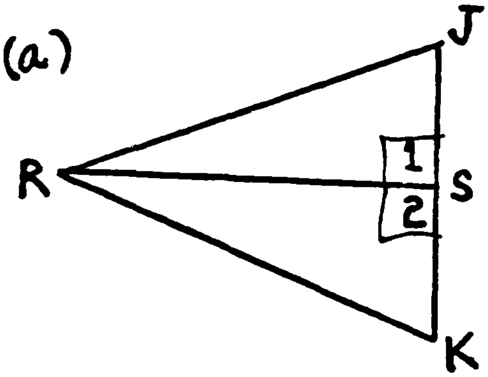
Discrimination provides a complementary process to generalization. It takes operators that are too general and thus are applying in incorrect situations and places restrictions on their range of applicability. If the operator to be discriminated is embodied as a production, discrimination adds an additional clause to restrict the range of situations where the production condition will match. ACT determines what additional clauses to add by comparing the difference between successful and unsuccessful application of the rule.

Figure 11 illustrates an analysis of a problem which led subject R to form a discrimination. In part (a) we have a representation of the problem and in part (b) we have indicated in search net form R's attempt to solve the problem. First he tried to use SSS, a method which had worked on a previous problem that had a great deal of superficial similarity to this problem. However, he was not able to get the sides $\overline{RK}$ and $\overline{RS}$ congruent. Then he switched to SAS, the other method he had at the time for proving triangles congruent. Interestingly, it was only in the context of this goal that he recognized the right angles were congruent. After he had finished with this problem, he verbally announced the rules to use SSS only if there was no angle mentioned. This can be seen to be the product of discrimination. The "don't use SSS if angle" comes from a comparison of the previous problem in which no angle was mentioned with the current problem that did mention angles.

Unfortunately, this is almost the only example of discrimination that we have been able to observe in R's protocols and it is the best. As in the case of generalization we feel this is because the text does not juxtapose many examples that offer opportunities for discrimination. Again subjectively, we feel that we as "experts" work with many discriminated operators but have no systematic data as to their origin.

ACT's generalization and discrimination processes were described in considerable detail in Anderson, Kline, and Beasley (1979). There we were concerned with showing how they applied in modelling the acquisition of category schema or prototypes. That data provided pretty strong evidence for the ACT mechanisms. It is partly the success of that enterprise that leads us to believe they play an important role in the development of expertise in geometry proof generation. Basically,

(a)

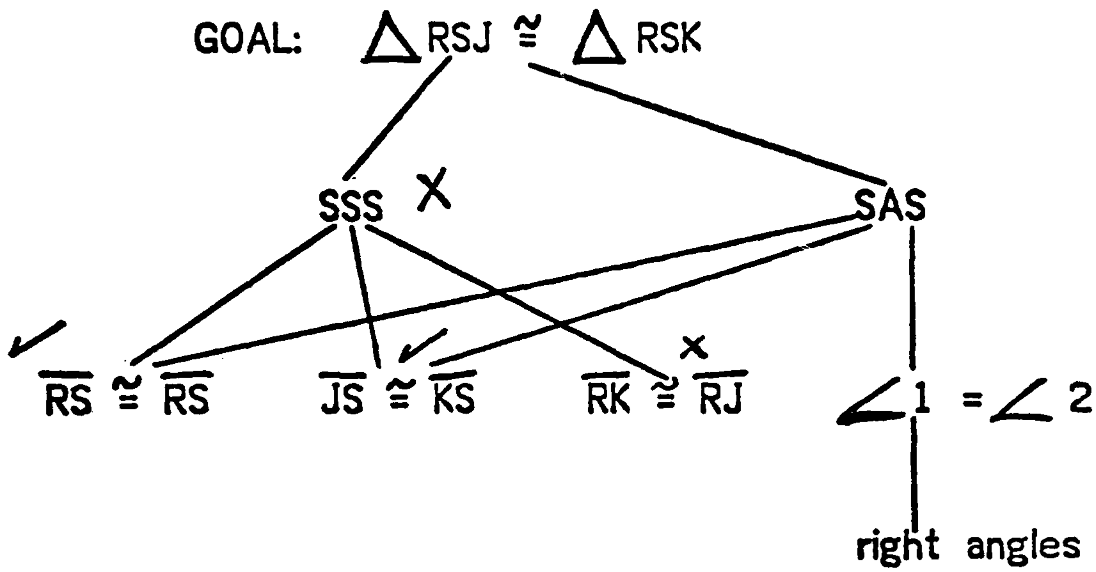


Given: $\angle 1$ and $\angle 2$ are right angles

$\overline{JS} \cong \overline{KS}$

Prove: $\triangle RSJ \cong \triangle RSK$

(b)



IF the goal is to prove $\triangle XYZ \cong \triangle UVW$ and no angles are mentioned

THEN try to prove this by means of SSS

IF the goal is to prove $\triangle XYZ \cong \triangle UVW$ and an angle is mentioned

THEN try to prove this by means of SAS

Figure 11

the claim is that students develop from examples schemata for when various proof methods are appropriate just as they develop schemata for what are examples of categories.

Composition

We feel that composition has an important role to play in forming multiple operator sequences just as it played an important role in the initial proceduralization of operators. Figure 12 illustrates an example where composition can apply. The first production to apply in solving this problem would be:

P24: IF the goal is to prove $\angle X \cong \angle U$
and $\angle X$ is part of $\triangle XYZ$
and $\angle U$ is part of $\triangle UVW$
THEN the subgoal is to prove $\triangle XYZ \cong \triangle UVW$

This production would set as a subgoal to prove $\triangle ABC \cong \triangle DBC$. At this point the following *production* might apply:

P25: IF the goal is to prove $\triangle XYZ \cong \triangle UVW$
and $\overline{XY} \cong \overline{UV}$
and $\overline{ZX} \cong \overline{WU}$
THEN the subgoal is to prove $\overline{YZ} \cong \overline{VW}$

This production, applied to the situation in Figure 12, would set as the subgoal to prove $\overline{BC} \cong \overline{BC}$ as a step on the way to using SSS. At this point the following production would apply:

P26: IF the goal is to prove $\overline{XY} \cong \overline{XY}$
THEN this may be concluded by reflexivity

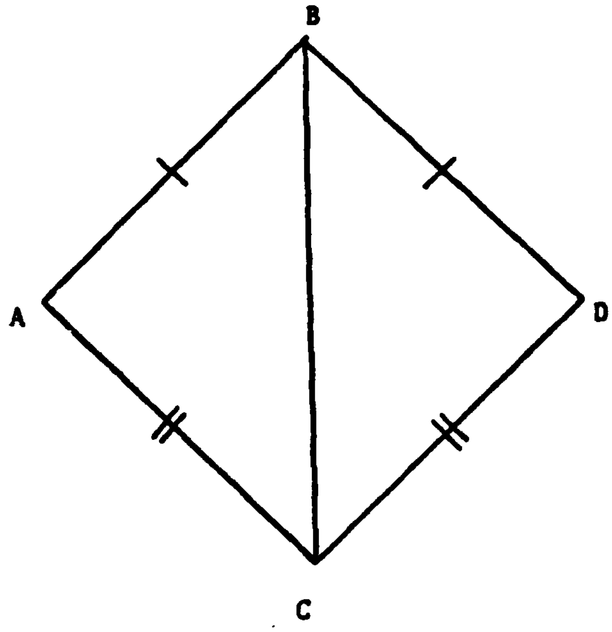
This production would add $\overline{BC} \cong \overline{BC}$ and allow the following production to apply:

P27: IF the goal is to prove $\triangle XYZ \cong \triangle UVW$
and $\overline{XY} \cong \overline{UV}$
and $\overline{YZ} \cong \overline{VW}$
and $\overline{ZX} \cong \overline{WU}$
THEN the goal may be concluded by SSS

where $\overline{XY} = \overline{AB}$, $\overline{UV} = \overline{DB}$, $\overline{YZ} = \overline{BC}$, $\overline{VW} = \overline{BC}$, $\overline{ZX} = \overline{CA}$, and $\overline{WU} = \overline{CD}$. This adds the information that $\triangle ABC \cong \triangle DBC$. Finally, the following production will apply which recognizes that the to-be-proven conclusion is now established:

P28: IF the goal to prove $\angle X \cong \angle U$
and $\triangle XYZ \cong \triangle UVW$
THEN the goal may be concluded because of congruent parts of congruent triangles

The composition process operating on this sequence of productions, would eventually produce a production of the form:



GIVEN: $\overline{AB} \cong \overline{DB}$
$\overline{CA} \cong \overline{CD}$

PROVE: $\angle A \cong \angle D$

Figure 12

P29: IF the goal is to prove $\angle A \cong \angle D$
and $\angle A$ is part of $\triangle ABC$
and $\angle D$ is part of $\triangle DBC$
and $\overline{AB} \cong \overline{DB}$
and $\overline{CA} \cong \overline{DC}$
THEN conclude $\overline{AB} \cong \overline{AB}$ by reflexivity
and conclude $\triangle ABC \cong \triangle DBC$ by SSS
and conclude the goal because of congruent parts of congruent triangles

The variables in this production have been named to correspond to the terms in Figure 12 for purposes of readability. This production would immediately recognize the solution to a problem like that in Figure 12. Another feature of composition, illustrated in this example, is that it transforms what had been a basically working backward solution to the problem into something much more of the character of working forward. Indeed, all the methods that we have discussed for tuning search operators, to the extent that they put into the conditions additional tests for applicability and into the action additional inferences, have the effect of converting working backward into working forward. Larkin, McDermott, Simon, and Simon (in press) have commented on this same transformation in the character of physics problem solving with the development of expertise.

Search Control

Our discussion of generalization, discrimination, and composition was mainly focused on how they can make reasoning forward or reasoning backward more efficient. In our discussion of composition we did show how putting a working forward flavor into operators that were basically working backward. However, another potential for these tuning processes is that they change the balance between working forward and working backward. In situations where working forward tends to be successful many tuned operators will be formed to promote this kind of proof development. In situations where working forward has proven not successful, the working forward operators will be discriminated against leaving more capacity for working backward. A similar selection will be applied to the working backward operators. There was no visible evidence in R's protocols for a development in the balance between working forward and working backward. However, another point of difference between ourselves and R was that our choices tend to be more apt about when to pursue forward or backward inferencing.

Summary of Geometry Learning

Figure 13 provides a summary of what we think the student's progress is as he gathers more experience and becomes more expert at generating proofs in geometry. There are two initial sources of information. One, labeled *rules*, refers to the postulates, theorems, and definitions that he reads in the text instructions. The second source, labeled examples, are the proof problems which are given as examples and other problems which he has solved himself. The rules are declaratively encoded into a schema-like form to which general problem solving productions can apply. As we discussed, the rules in this form are applied in a piecemeal way. We also discussed the twin processes of knowledge compilation, composition and instantiation, by which these rules can eventually achieve a procedural form in which each rule is embodied by a production.

The examples can be used through analogy to guide the solution of prolems. We characterize this process as basically top-down in which the student tries to adapt the solution from one problem to another. We think that the solution by analogy involves interpretative processing of the examples much as the initial use of the general rules. However, as noted, specific examples are very limited in their range of applicability. In Figure 13 we indicate that the processes of compilation and generalization applied to these examples can lead the student to the same kind of general, proceduralized, unitary operators as can compilation applied to the rules. To the extent that generalization leaves in features of the original problems, the operators from this source might not be as general as the operators derived directly from the rules, but rather will remain tuned to specific problem characteristics.

Finally, we indicate that the processes of discrimination and composition create larger multiple-inference operators which are much more discriminant in their range of applicability. In the extreme we get special rules that outline full proof trees for certain kinds of problems. The character of these operators is, as we have noted, working forward more than working backward. To the extent that new problems fit the specifications of these advanced operators, solution will be quick and efficient. However, to the extent new problems pose novel configurations of features not covered by the advanced operators the student will have to fall back to the slower and more general operators for

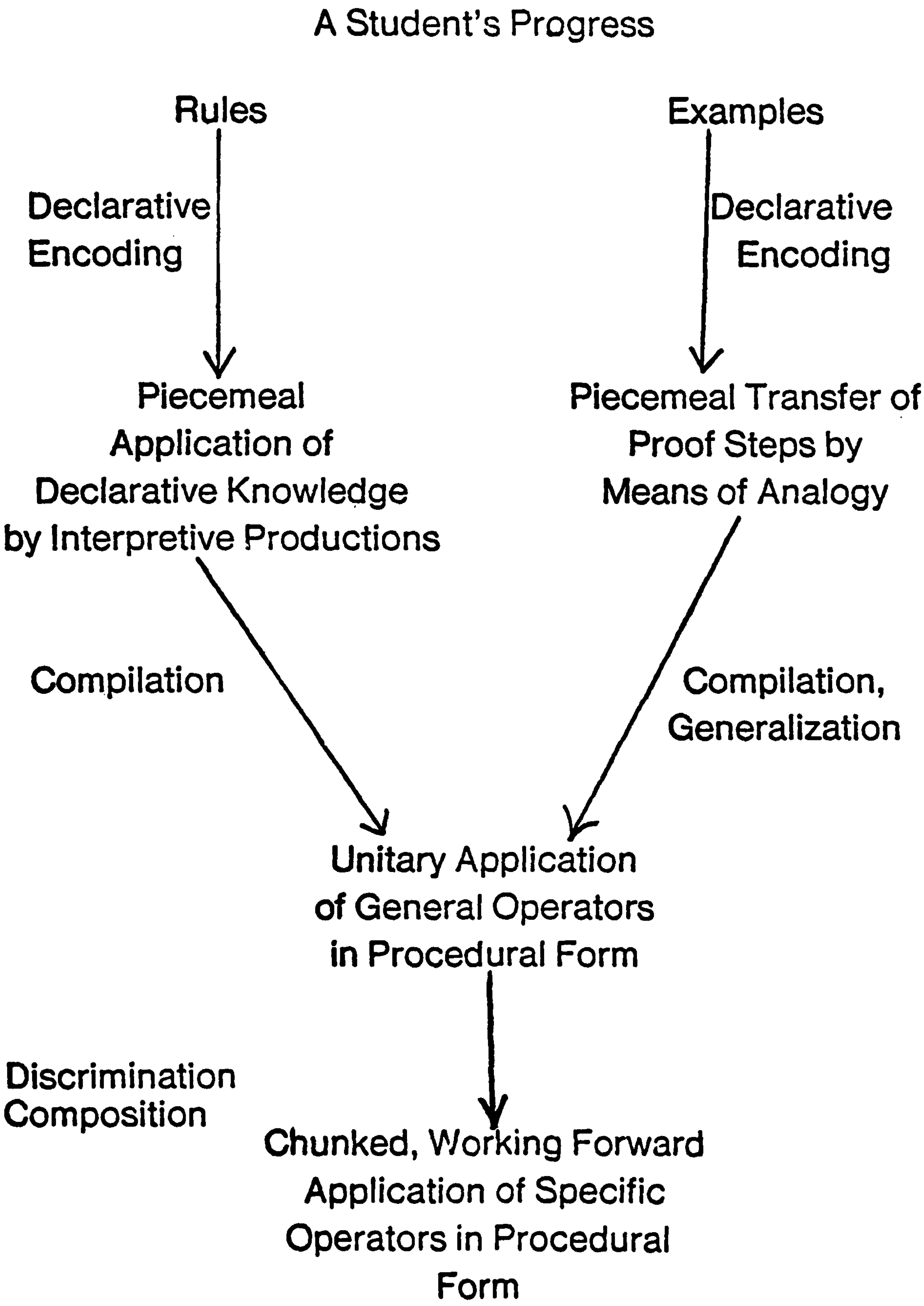

A Student's Progress
Rules
Examples
Declarative Encoding
Declarative Encoding
Piecemeal Application of Declarative Knowledge by Interpretive Productions
Piecemeal Transfer of Proof Steps by Means of Analogy
Compilation
Compilation, Generalization
Unitary Application of General Operators in Procedural Form
Discrimination Composition
Chunked, Working Forward Application of Specific Operators in Procedural Form

Figure 13

working backwards. We certainly notice this variation in our own behavior as "experts" depending on how unique a geometry problem is. Our view of expertise, then, is very much the one that was developed for chess (refs); that is, experts in geometry proof generation have simply encoded many special case rules.

Recommendations for the Teaching of Geometry

One would hope that a good task analysis and a good learning analysis would have some recommendations to make for instruction. We would like to conclude this paper with a brief statement of what we think the pedagogical implications are of the theory that was developed here.

One clear implication of the theory that we developed was that the role of planning in proof generation should be emphasized. This is both because it is critical to proof generation and it is one skill that one might hope would generalize to other problem-solving skills. Related to this is the importance of the proof tree as a hierarchical structure relating the to-be-proven statement to the givens. Unfortunately, the nature of two-column geometry proofs leads to a line-by-line approach to teaching proof generation -- an approach that is exacerbated by the tendency in textbooks to use reason-giving exercises as introductions to proof generation. We suggest that proofs be generated in a hierarchical form as a step to make clear the hierarchical plan that underlies the proof. This, by the way, is a suggestion that has also been offered in the standard literature on geometry instruction (ref).

The perceptual difficulties of seeing certain kinds of relationships like vertical angles or shared angles suggests perceptual training in which students get special practice at seeing various relationships. This is a recommendation that is already incorporated at least to a degree in standard texts.

If our analysis of the declarative encoding of postulates and theorems into background, hypotheses, and conclusion is correct, then it is clear that the statement of the theorems should be formulated in such a way that identification of the three components is easy. Students are given practice at analyzing such statements into if-then form, thereby identifying hypothesis and conclusion. However, identification of background is obscure and not explicit. Usually, it is left to be

inferred from the diagram. Most of the problems our subject R has in encoding were because it was just too difficult to infer what was implicit in the diagram. For instance, this is why he failed to properly represent the included angle relationship required in the SAS postulate.

It is also important that the student commit firmly to memory the definitions, postulates, and theorems of the text. At least in some circles this is standard pedagogical wisdom, but we ignored it in our instruction of R and we think it was to his detriment. He often referred back in the text to recall a postulate or definition and for some he never did manage to commit them to memory. Recall that the instantiation component of procedural compilation requires that the knowledge be committed to long term memory.

There are a number of recommendations to be made related to the process of tuning the operators so that they apply in more opportune situations. It is not clear, however, how important it is to get the student to the point of being capable of finely tuned search for geometry proofs. The skill is quite geometry-specific. Some people would argue that the purpose of geometry instruction is to teach problem solving, general proof knowledge and skill, and basic facts of geometry but not to make students expert at generating geometry proofs.

However, if one's goal is to create experts at geometry proofs, then the first remark to be made is that drill and practice, that old technique of education, is critical. The student needs to get the examples from which generalization, discrimination, and composition can tune the operators. However, it also follows that the examples on which the student works are critical. Generalization requires that the student see together examples that instantiate some common, underlying operators. Similarly, discrimination requires that the student see examples together that display critical minimal contrasts. Finally, composition requires that we place together examples that exemplify an important sequence of proof steps.

As we mentioned in our discussion of generalization, discrimination, and composition, standard texts score quite poorly in creating these critical juxtapositions. This is largely because the number of exercises are few relative to the number required to reliably create the generalizations, discriminations, and compositions required. (This is exacerbated by practices of only assigning

students every other exercise.) In order to cover all the basics there is not much opportunity to create critical redundancies.

It might be possible to impart to students the tuned operators more efficiently than by induction from example. Induction of rules from example is required when the teacher does not really know the rules or how to impart them to students. It might be possible with careful task analysis to identify what the optimal form of the operators is and to directly instruct students as to these operators. However, they would still need considerable practice to achieve procedural form.

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